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A SIMILARITY SOLUTION FOR COMBINED FORCED AND FREE CONVECTION FLOW OVER A HORIZONTAL PLATE

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Abstract—The effect of buoyancy forces on the steady, laminar, plane flow over a horizontal plate is investigated within the framework of a first-order boundary layer theory, taking into account the hydrostatic pressure variation normal to the plate. An exact similarity solution is given for the case of a wall temperature that is inversely proportional to the square root of the distance from the leading edge. Remarkably, such a solution does not exist if the buoyancy parameter is smaller than a certain critical value, which is negative (decelerated flow) but not yet small enough to satisfy the separation criterion of vanishing shear stress at the wall. Although the wall temperature is different from the free-stream temperature, there is no local heat transfer at the wall except in the singular point at the leading edge. The total heat transfer is finite, independent of the plate length, and is calculated by applying the heat flux equation. The displacement thickness is also given. It is negative if the plate is heated sufficiently strongly.

NOMENCLATURE

Ar, Archimedes number, equation (4);
c_p, specific heat capacity;
f, reduced stream function, equation (10);
g, gravity constant;
K, buoyancy parameter, $Ar/\sqrt{Re}$;
L, value of x where $T_w - T_\infty = T^*$;
l, plate length;
O, Landau's order symbol;
P, dimensionless pressure difference, equation (2);
Pr, Prandtl number;
p, pressure;
Q, heat exchanged between plate and fluid;
Re, Reynolds number, $u_\infty L/\nu$;
St, Stanton number, equation (17);
T, temperature;
T^*, characteristic temperature difference between plate and free stream;
U, V, dimensionless velocity components, equation (2);
u, v, velocity components in x,y-directions;
X, Y, dimensionless Cartesian coordinates, equation (2);
x, y, Cartesian coordinates.

Greek symbols

β, thermal expansivity;
δ^*, displacement thickness;
η, similarity variable, equation (10);
θ, $(T-T_\infty)/T^*$;
ϑ, reduced temperature difference, equation (10);
ν, kinematic viscosity;
ρ, density;
ψ, dimensionless stream function, equation (6).

Subscripts

w, at the wall;
∞, in the free stream.

1. INTRODUCTION

IN CLASSICAL boundary layer theory, body forces such as buoyancy forces are taken into account only with respect to their tangential component. Force components normal to the surface are neglected as higher-order terms, with the result of no pressure variation across the boundary layer.

There are, however, flow phenomena which cannot adequately be described by means of the classical boundary layer equations. Examples are the bending of a hot, non-vertical jet in a gravity field, and the influence of buoyancy forces on the flow over a heated or cooled horizontal plate. In the latter problem the body forces normal to the surface may even result in separation of the boundary layer flow.

A possible way of attacking problems of that kind is provided by second-order boundary layer theory [1]. The normal component of the buoyancy force, like streamline curvature and displacement, contributes to a second-order pressure gradient normal to the surface. Obviously, this perturbation theory with respect to large Reynolds numbers is only correct if the buoyancy term remains small in comparison with the first-order terms. Since the buoyancy term is connected with a dimensionless parameter (Froude number or Archimedes number), and since this parameter is independent of the Reynolds number, the assumption of a sufficiently small buoyancy term is not always satisfied. Furthermore, for free jets as well as for the boundary-layer at a horizontal plate, the relative importance of buoyancy forces as compared to inertia forces typically increases with increasing distance from the orifice or leading edge, respectively. As a consequence, the perturbation procedure breaks down at a certain distance downstream, as can be seen, for instance, by inspection of the solutions presented in [1].

In the particular case of a curved free jet the complications involved with the second-order-boundary-layer theory can be evaded by applying the conservation equations in integral form and using *ad hoc* assumptions for the velocity and temperature profiles (integral method [2, 3]). As far as the boundary layer flow over a horizontal plate is concerned, however, the pressure variation normal to the wall is essential for a representation of the buoyancy effects. If, in this case, the buoyancy term is large as compared with the curvature term, a modification of the first-order boundary layer theory is necessary and possible.

Approximate solutions of the modified first-order boundary layer equations have already been obtained by several investigators. Perturbation series in terms of the distance from the leading edge [4–7] are valid only for small buoyancy effects and are therefore subject to the same limitations as the results of the second-order boundary layer theory [1]. Hieber's work [7] contains also expansions for very large distances from the leading edge (buoyancy dominated region). The limiting cases of very small and very large Prandtl numbers have been studied by the method of matched asymptotic expansions [8]. Most recently, approximate solutions for quite a large range of the buoyancy parameter have been obtained by local similarity and local non-similarity methods [9]. Apparently, exact solutions are not available so far. It is the purpose of this investigation to fill that gap and give an exact similarity solution.

2. BASIC EQUATIONS

Consider a horizontal flat plate aligned parallel to a uniform free stream with velocity u_∞, density ρ_∞, and temperature T_∞. The plate is maintained at a certain temperature T_w which may depend on the longitudinal coordinate x according to the relation

$$T_w - T_\infty = T^*\theta_w(x/L), \quad \text{with } \theta_w(1) = 1. \tag{1}$$

T^* represents a characteristic temperature difference between plate and free stream, and L is the value of the x-coordinate where $T_w - T_\infty = T^*$. If the wall temperature is constant, L can be chosen arbitrarily.

The flow over the plate is considered to be plane, laminar, and steady. Constant transport coefficients are assumed, and the Boussinesq approximation is applied.

A Cartesian coordinate system x, y is used with the origin at the leading edge of the plate, and the velocity components in x- and y-direction are denoted by u and v, respectively. Furthermore, the dimensionless variables,

$$X = x/L; \quad Y = \sqrt{Re}\, y/L;$$
$$U = u/u_\infty; \quad V = \sqrt{Re}\, v/u_\infty; \tag{2}$$
$$\theta = (T - T_\infty)/T^*; \quad P = (p - p_\infty)/\rho_\infty u_\infty^2,$$

are introduced, where $Re = u_\infty L/\nu$ is the Reynolds number and the pressure is referred to p_∞, i.e. the hydrostatic pressure in the undisturbed fluid. Applying now the boundary layer approximation in the common manner but with the modification that the pressure varies across the boundary layer and is equal to the hydrostatic pressure in the temperature-disturbed fluid, the following boundary layer equations are obtained:

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0; \tag{3a}$$

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial Y} + \frac{\partial P}{\partial X} = \frac{\partial^2 U}{\partial Y^2}; \tag{3b}$$

$$\frac{\partial P}{\partial Y} = K\theta; \tag{3c}$$

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2 \theta}{\partial Y^2}. \tag{3d}$$

Pr stands for the Prandtl number, and the buoyancy parameter K is related to the Archimedes number Ar according to

$$K = Ar/\sqrt{Re}, \quad Ar = gL\beta_\infty T^*/u_\infty^2, \tag{4}$$

where g is the gravity acceleration, and β_∞ is the thermal expansivity of the fluid in the undisturbed state. The signs in equation (3c) have been chosen for the flow *above* the plate; if the flow below the plate were considered K should be replaced by $-K$ in equation (3c).

The boundary conditions in dimensionless form are

$$U = V = 0,\ \theta = \theta_w(X), \quad \text{on } Y = 0,\ X > 0; \tag{5a}$$
$$U = 1,\ \theta = P = 0, \quad \text{as } Y \to \infty. \tag{5b}$$

Boundary layer theory is based on an asymptotic expansion for $Re \to \infty$. Since Ar is an independent parameter, the following cases are to be distinguished:

I. $Ar = O(1)$: The right-hand side of equation (3c) is of order $O(Re^{-1/2})$, i.e. of the same order of magnitude as the second-order terms stream-line curvature and displacement. Hence buoyancy effects can be—and should be—included in a second-order boundary layer theory; see [1].

II. $Ar \to \infty$ such that $K \to 0$: The right-hand side of equation (3c) is very large compared with the second-order boundary layer terms but very small compared with the first-order terms. This case is included in the second-order boundary layer theory [1].

III. $Ar \to \infty$ such that $K = O(1)$: The right-hand side of equation (3c) is part of a first-order boundary layer theory. It is this case which is to be considered in the present paper.

IV. $Ar \to \infty$ such that $K \to \infty$: Free convection is dominant. The correct solution has been given in [10].

If the wall temperature is constant, the boundary conditions do not contain any characteristic length. In this special case the length L can be chosen such that $K = 1$. With this choice of L, the regime of equal importance of free and forced convection is given by $X = O(1)$. The case of constant heat flux at the plate surface can be treated in an analogous manner. Since neither of the two problems is accessible to an exact similarity solution we will not go into further details here.

3. SIMILARITY SOLUTION

The continuity equation (3a) is satisfied by introducing a stream function ψ with:

$$U = \psi_Y, \quad V = -\psi_X. \tag{6}$$

Subscripts X and Y indicate partial derivatives. Furthermore the pressure is eliminated by integrating the normal momentum equation (3c) with respect to Y and using the boundary condition $P = 0$ as $Y \to \infty$. From the tangential momentum equation (3b) and the energy equation (3d) we obtain the following system of equations for ψ and θ:

$$\psi_Y\psi_{XY} - \psi_X\psi_{YY} - K\int_Y^\infty \theta_X \, dY = \psi_{YYY}; \tag{7a}$$

$$\psi_Y\theta_X - \psi_X\theta_Y = \frac{1}{Pr}\theta_{YY}. \tag{7b}$$

The boundary conditions are:

$$\psi = \psi_Y = 0, \ \theta = \theta_w(X), \quad \text{on } Y = 0, X > 0; \tag{8a}$$

$$\psi_Y = 1, \theta = 0, \quad \text{as } Y \to \infty. \tag{8b}$$

Looking now for similarity solutions by one of the well-known methods (see e.g. [11, 12]) we find that a similarity solution is possible if the wall temperature distribution is of the form

$$\theta_w = X^{-1/2}. \tag{9}$$

In this case the similarity transformation

$$\eta = YX^{-1/2}; \qquad \psi = X^{1/2}f(\eta), \quad \theta = \theta_w\vartheta(\eta), \tag{10}$$

yields the system of equations

$$2f''' + ff'' + K\eta\vartheta = 0; \tag{11a}$$

$$\frac{2}{Pr}\vartheta'' + f\vartheta' + f'\vartheta = 0, \tag{11b}$$

where primes denote differentiation with respect to the similarity coordinate η. From the boundary conditions (8a) and (8b) we obtain

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1; \tag{12a}$$

$$\vartheta(0) = 1; \tag{12b}$$

$$\vartheta(\infty) = 0. \tag{12c}$$

Note that not only the partial derivatives have been reduced to ordinary derivatives but also the integral appearing in equation (7a) has been eliminated.

Equation (11b) can be integrated at once. Using the boundary condition (12c) we obtain

$$\frac{2}{Pr}\vartheta' + f\vartheta = 0. \tag{13}$$

This first-order differential equation can be formally integrated to give

$$\vartheta = \exp\left(-\frac{Pr}{2}\int_0^\eta f \, d\eta\right), \tag{14}$$

where the boundary condition (12b) has already been satisfied. From the numerical point of view, however, it is preferable to use equation (13) instead of equation (14).

4. NUMERICAL RESULTS AND DISCUSSION

The system of ordinary differential equations (11a) and (13) subject to the boundary conditions (12a) and (12b) has been solved numerically by two different methods. Figures 1 to 3 show results which have been obtained by the common shooting method. The case $Pr = 1$ has also been investigated by the method of parametric differentiation (see e.g. [12, 13]). In general, the results of both methods agreed satisfactorily, but when using the method of parametric differentiation for negative values of K (corresponding to an adverse pressure gradient) an extremely small step size was necessary in order to obtain the required accuracy.

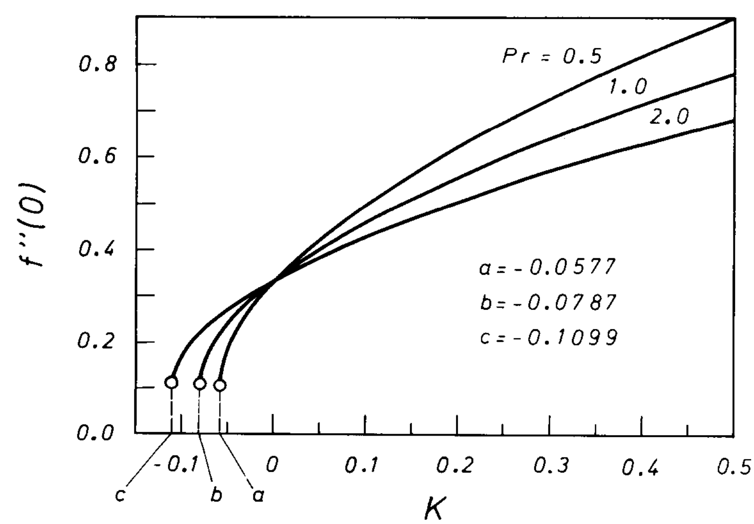

FIG. 1. Dimensionless wall shear stress, $f''(0)$, as a function of the buoyancy parameter $K = Ar/\sqrt{Re}$ for various Prandtl numbers. For $Pr = 0.5$, 1.0, and 2.0, respectively, there are no solutions for K-values smaller than the critical values -0.0577, -0.0787, and -0.1099.

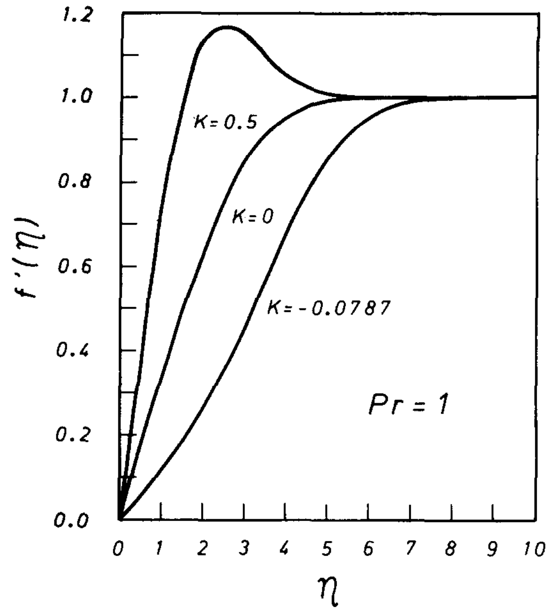

FIG. 2. Tangential velocity profiles as a function of the similarity variable η for various values of the buoyancy parameter K.

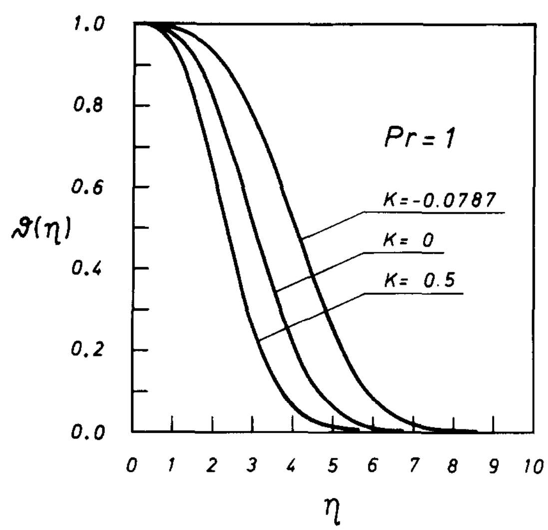

FIG. 3. Temperature profiles as a function of the similarity variable η for various values of the buoyancy parameter K.

Figure 1 shows $f''(0)$, which is proportional to the wall shear stress, as a function of the buoyancy parameter K for three different Prandtl numbers. At $K = 0$ (no buoyancy) the Blasius value $f''(0) = 0.332$ is obtained, of course. If $K > 0$ (i.e. plate temperature larger than free-stream temperature) there is a favorable pressure gradient above the plate due to buoyancy effects and the wall shear stress is larger than in the non-buoyant case. If $K < 0$ (i.e. plate temperature smaller than free-stream temperature) the opposite is true.

One might expect that at a certain negative value of K the separation condition $f''(0) = 0$ would be satisfied. This is not the case, however. It can be seen from Fig. 1 that below a certain critical value of K there is no similarity solution of the boundary layer equations as used in this paper. The critical values depend on the Prandtl number and are, according to our calculations, given by $K = -0.0577$, -0.0787, and -0.1099 for $Pr = 0.5$, 1.0, and 2.0, respectively. It appears that these results of an exact solution explain why convergence of the truncation method used in [9] could not be attained for $Gr_x/Re_x^{5/2}$ smaller than -0.03, where Gr_x and Re_x are, respectively, the local Grashof and Reynolds numbers.

Typical velocity and temperature profiles are shown in Figs. 2 and 3, respectively. The Prandtl number is equal to 1 in all cases. As far as the negative value of K is concerned, the velocity and temperature profiles just before breakdown of the similarity solution are given.

It is interesting to note that $\vartheta'(0) = 0$ for all values of K, cf. Fig. 3. This result, which follows directly from equation (13) together with the boundary condition (12a), indicates that there is no local heat transfer at the plate surface for all $X > 0$. Nevertheless, although dissipation has been neglected, the temperature of the fluid is changed during the flow process. The paradox is resolved by recalling that the similarity solution requires a singular behaviour of the wall temperature at $X = 0$, cf. equation (9). Thus all the heat necessary to change the fluid temperature must be transferred in the singular point $X = 0$, which is the leading edge of the plate.

Consider a plate of length l (not to be confused with L, i.e. the characteristic length of the wall-temperature distribution). In order to circumvent the difficulties linked to the singularity at the leading edge the total heat transfer Q_w is determined with the help of the heat flux equation

$$Q_w = \rho_\infty c_p \int_0^\infty [(T - T_\infty)u]_{x=l}\, dy, \tag{15}$$

where c_p is the specific heat capacity at constant pressure. Introducing the dimensionless variables of equation (2) and applying the similarity transformation (10), we obtain the relation

$$\sqrt{Re}\, St = \int_0^\infty \vartheta f'\, d\eta = \text{const}, \tag{16}$$

where the Stanton number

$$St = Q_w/\rho_\infty u_\infty c_p T^* L, \tag{17}$$

is referred to the characteristic length of the wall-temperature distribution. According to equation (16) the Stanton number is independent of the plate length l, thereby confirming the statement that the total heat transfer takes place at the leading edge. Numerical results for the Stanton number are shown in Fig. 4.

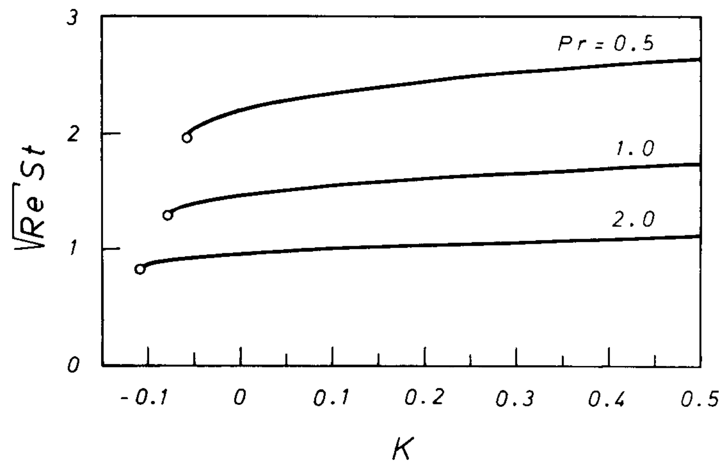

FIG. 4. Total heat transfer at the plate (Stanton number St) as a function of the buoyancy parameter K for various Prandtl numbers.

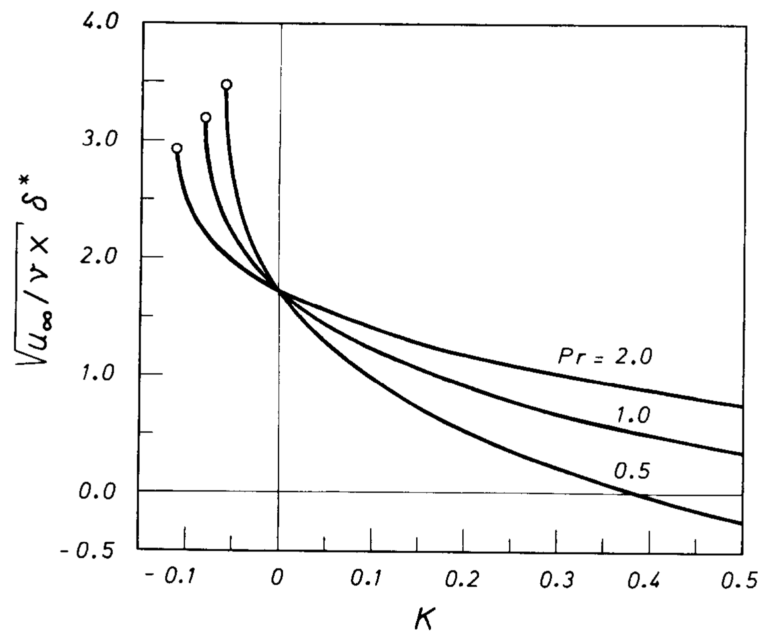

FIG. 5. Effect of buoyancy parameter K and Prandtl number Pr on the displacement thickness δ^*.

Finally, the displacement thickness δ^* can be calculated from the relations

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy = \sqrt{\frac{\nu x}{u_\infty}} \int_0^\infty [1 - f'(\eta)]\, d\eta$$

$$= \sqrt{\frac{\nu x}{u_\infty}} \lim_{\eta \to \infty} [\eta - f(\eta)]. \quad (18)$$

The limit value is given in Fig. 5. It is interesting to note that the displacement thickness is negative for relatively large values of the buoyancy parameter K, i.e. if the plate is heated sufficiently strongly. This is a consequence of local velocities that are larger than the free stream velocity, cf. the velocity profile for $K = 0.5$ in Fig. 2.

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Note added in proof—At the suggestion of participants of the GAMM-Conference 1979 in Wiesbaden, Germany, it was found that the solution is not unique for values of K slightly larger than the critical value. A second branch of the solution shown in Fig. 1 originates in the critical point and, for $Pr = 1$, intersects the line $f''(0) = 0$ at $K = -0.050$ (separation point). Whether the second branch is a stable solution and can experimentally be realized at a plate that is strongly heated at the leading edge but otherwise isolated needs further investigation.

UNE SOLUTION DE SIMILITUDE POUR LA CONVECTION MIXTE SUR UNE PLAQUE HORIZONTALE

Résumé—On étudie l'effet des forces d'Archimède sur l'écoulement permanent, laminaire et plan sur une plaque horizontale, dans le cadre d'une théorie de couche limite du premier ordre, prenant en compte la variation de pression hydrostatique normalement à la plaque. Une solution de similitude exacte est donnée dans le cas d'une température de paroi inversement proportionnelle à la racine carrée de la distance au bord d'attaque. Une réelle solution n'existe pas si le paramètre de convection naturelle est inférieur à une valeur critique laquelle est négative (écoulement décéléré), mais pas suffisamment petit pour satisfaire le critère de séparation, d'annulation de la contrainte à la paroi. Bien que la température pariétale est différente de celle de l'écoulement libre, il n'y a pas de transfert thermique local à la paroi, sauf au point singulier du bord d'attaque. Le transfert thermique total est fini, indépendant de la longueur de la plaque, et il est calculé par application de l'équation du flux thermique. On donne aussi l'épaisseur de déplacement. Elle est negative si la plaque est chauffé suffisamment fortement.

EINE ÄHNLICHKEITSLÖSUNG FÜR DIE STRÖMUNG LÄNGS EINER HORIZONTALEN PLATTE BEI KOMBINIERTER ERZWUNGENER UND FREIER KONVEKTION

Zusammenfassung—Der Einfluß statischer Auftriebskräfte auf die stationäre, laminare, ebene Strömung längs einer horizontalen Platte wird im Rahmen einer Grenzschichttheorie erster Ordnung untersucht, wobei die hydrostatische Druckänderung normal zur Platte zu berücksichtigen ist. Eine exakte Ähnlichkeitslösung wird für den Fall angegeben, daß sich die Wandtemperatur umgekehrt proportional zur Wurzel aus dem Abstand von der Vorderkante ändert. Bemerkenswerterweise existiert eine solche Lösung nicht, wenn der Auftriebsparameter kleiner als ein bestimmter kritischer Wert ist; dieser kritische Wert ist negativ (verzögerte Strömung), aber noch nicht hinreichend klein, um das Ablösekriterium verschwindender Wandschunspannung zu erfüllen. Obwohl sich die Wandtemperatur von der Temperatur der ungestörten Strömung unterscheidet, gibt es keinen örtlichen Wärmeübergang an der Wand, ausgenommen den singulären Punkt an der Vorderkante. Der gesamte Wärmeübergang ist endlich, unabhängig von der Plattenlänge und wird mittels der Wärmestromgleichung berechnet. Auch die Verdrängungsdicke wird angegeben. Sie ist negativ, wenn die Platte hinreichend stark geheizt wird.

АВТОМОДЕЛЬНОЕ РЕШЕНИЕ ЗАДАЧИ О СМЕШАННОЙ СВОБОДНОЙ И ВЫНУЖДЕННОЙ КОНВЕКЦИИ НА ГОРИЗОНТАЛЬНОЙ ПЛАСТИНЕ

Аннотация — В приближении теории пограничного слоя первого порядка исследуется влияние подъёмных сил на стационарное, ламинарное, плоское течение на горизонтальной пластине с учётом изменения гидростатического давления по нормали к пластине. Найдено точное автомодельное решение для случая, когда значение температуры стенки изменяется обратно пропорционально корню квадратному расстояния от передней кромки. Следует отметить, что такое решение не является справедливым в том случае, если величина свободноконвективного параметра меньше определенного критического значения (заторможенное течение), но не столь малым, чтобы удовлетворять критерию отрыва потока при стремящемся к нулю напряжении сдвига на стенке. Хотя температура стенки отлична от температуры свободного потока, локальный теплообмен на стенке отсутствует за исключением сингулярной точки на передней кромке. Величина суммарного теплообмена является конечной, не зависит от длины пластины и рассчитывается с помощью уравнения для теплового потока. Определена также толщина вытеснения. Эта величина является отрицательной в случае достаточно сильного нагрева.